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New Spinor Representation of Maxwell's Equations. I. Generalities

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A new representation of the electromagnetic field tensor has been found. In this representation it is shown that an intimate relationship exists between electromagnetism and spin; the duality rotation of the "already unified theory" is shown to coincide with the Touschek–Nishijima transformation of the theory of leptons. A nonlinear spinor equation equivalent to Maxwell's equations is deduced.

1. INTRODUCTION

In this paper a new spinor representation of Maxwell's equations is deduced. Spinor representation of Maxwell's equations has already been given by several authors (Laporte and Uhlenbeck, 1931; Oppenheimer, 1931; von Molière, 1949; Von Schubert, 1949; Ohmura, 1956; Good, 1957; Moses, 1958, 1959). However the present one differs completely from all the previous ones. The present representation shows a relationship between electromagnetic field and the spin of relativistic quantum mechanics, and moreover the duality rotation of Rainich (1925) and Misner and Wheeler (1957) is naturally identified with the Touschek (1957) and Nishijima (1957) transformation of the theory of leptons. These results indicate the possibility of an intimate correspondence between electromagnetism and relativistic quantum mechanics. This aspect of the problem will be investigated in detail in forthcoming papers. It also appears of interest that while the present representation of Maxwell's equations quadratically involves a spinor, typical of the electromagnetic field, it has been possible to deduce a

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single equation for such a spinor which is nonlinear and completely equivalent to Maxwell's equations.

2. A SPINOR REPRESENTATION OF MAXWELL'S EQUATIONS

The tensor form of Maxwell's equations is well known. Once we introduce the electromagnetic field tensor $F_{\mu\nu}$ as follows,

$$F_{\mu\nu} = -F_{\nu\mu} = A_{\mu,\nu} - A_{\nu,\mu} \tag{2.1}$$

Maxwell's equations read[2]

$$F^{\mu\nu}{}_{,\mu} = j^{\nu} \tag{2.2}$$

$$*F^{\mu\nu}{}_{,\mu} = 0 \tag{2.3}$$

where the covariant and contravariant components are related as follows:

$$F^{\mu\nu} = \eta^{\mu\sigma}\eta^{\nu\tau}F_{\sigma\tau} \tag{2.4}$$

$*F^{\mu\nu}$ is the dual of $F^{\mu\nu}$ defined by

$$*F^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\sigma\tau}\eta_{\sigma\alpha}\eta_{\tau\beta}F^{\alpha\beta} \tag{2.5}$$

with $\varepsilon^{\mu\nu\sigma\tau}$ the Ricci pseudotensor with entry $+1$ if the parity of the permutation $\mu\nu\sigma\tau$ of the indices $0, 1, 2, 3$ is even, and -1 if odd, and entry zero if two or more indices are equal, and with $\eta^{\mu\nu}$ the Minkowski metric tensor given by

$$\eta^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \qquad (\mu, \nu = 0-3) \tag{2.6}$$

It is well known (Messiah, 1966b, p. 908) that for any set of Dirac matrices γ which obey the anticommutation relations

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu} \tag{2.7}$$

[2]Hereafter we shall use the Einstein sum convention under which the sum is understood when two indices are repeated, and a comma followed by an index indicates the operation of partial derivative with respect to the variable with that index.

any spinor Ψ with four complex components is such that the two-indices system

$$F^{\mu\nu}=\tfrac{1}{2}\bar{\Psi}\gamma^{[\mu}\gamma^{\nu]}\Psi \tag{2.8}$$

transforms as an antisymmetric real tensor of rank 2.

Here we have indicated by $\gamma^{[\mu}\gamma^{\nu]}$ the following:

$$\gamma^{[\mu}\gamma^{\nu]}=\tfrac{1}{2}[\gamma^{\mu},\gamma^{\nu}]=\tfrac{1}{2}(\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\nu}) \tag{2.9}$$

and by $\bar{\Psi}$ the Dirac conjugate of the spinor Ψ, namely,

$$\bar{\Psi}=\Psi^{\dagger}\gamma^{0} \tag{2.10}$$

where $\Psi^{\dagger}$ is the Hermitian conjugate of the spinor Ψ.

Our objective is to demonstrate that any electromagnetic field tensor is susceptible of a representation (2.8). In the sense that for any electromagnetic field tensor $F^{\mu\nu}$ there exists at least a spinor Ψ such that equation (2.8) holds. In doing this we will use a theorem found by Rainich (1925) and revived by Misner and Wheeler (1957).

This theorem states that at any point of the four-dimensional Minkowski space any nonnull electromagnetic field can be reduced to an extremal field by a Lorentz transformation and a duality rotation.

Misner and Wheeler call an extremal field, a field for which the magnetic field **H** is zero and the electric field **E** is parallel to the x axis.

While the Lorentz transformation is quite well known, the duality rotation is not. It is, however, readily defined as the operation which brings any antisymmetric tensor $F^{\mu\nu}$ in a double-index system $\bar{F}^{\mu\nu}$ by means of the operation

$$\bar{F}^{\mu\nu}=F^{\mu\nu}\cos\alpha+{}^{*}F^{\mu\nu}\sin\alpha \tag{2.11}$$

The real parameter α is called by Misner and Wheeler the "complexion" of the field $F^{\mu\nu}$ in the given point. Once the proof of the validity of the representation (2.8) for any nonnull field $F^{\mu\nu}$ will be given, then from the following identity

$$\varepsilon^{\mu\nu\sigma\tau}\eta_{\sigma\alpha}\eta_{\tau\beta}\gamma^{[\alpha}\gamma^{\beta]}=2\gamma^{5}\gamma^{[\mu}\gamma^{\nu]} \tag{2.12}$$

with

$$\gamma^{5}=\gamma^{0}\gamma^{1}\gamma^{2}\gamma^{3} \tag{2.13}$$

and hence

$$(\gamma^5)^2 = -1, \tag{2.14}$$

and

$$\gamma^{5\dagger} = -\gamma^5 \tag{2.15}$$

where one uses the common representation of the γ matrices

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

$$\gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \quad \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{2.16}$$

with

$$(\gamma^0)^2 = 1, \qquad (\gamma^k)^2 = -1 \qquad (k = 1,2,3) \tag{2.17}$$

one has

$$*F^{\mu\nu} = \frac{i}{2} \overline{\Psi} \gamma^5 \gamma^{[\mu} \gamma^{\nu]} \Psi \tag{2.18}$$

On the other hand the matrices $(i/2)\gamma^{[\mu}\gamma^{\nu]}$ are nothing but the matrices of the matrix representation of the spin operator $S^{\mu\nu}$ (Messiah, 1966b, p. 905)

$$S^{\mu\nu} = \frac{i}{2} \gamma^{[\mu} \gamma^{\nu]} \tag{2.19}$$

and equivalently one has

$$F^{\mu\nu} = \overline{\Psi} S^{\mu\nu} \Psi \tag{2.20}$$

$$*F^{\mu\nu} = \overline{\Psi} \gamma^5 S^{\mu\nu} \Psi \tag{2.21}$$

And Maxwell's equations (2.2) and (2.3) read

$$(\overline{\Psi} S^{\mu\nu} \Psi)_{,\mu} = j^\nu \tag{2.22}$$

$$(\overline{\Psi} \gamma^5 S^{\mu\nu} \Psi)_{,\mu} = 0 \tag{2.23}$$

The energy-momentum tensor T_ν^μ is readily calculated in the spinor representation. In fact the familiar definition

$$T_\nu^\mu = F^{\mu\sigma}F_{\nu\sigma} - \tfrac{1}{4}\delta_\nu^\mu F^{\sigma\rho}F_{\sigma\rho}$$

when is written in the representation (2.20) becomes

$$T_\nu^\mu = (\overline{\Psi}S^{\mu\sigma}\Psi)(\overline{\Psi}S_{\nu\sigma}\Psi) - \tfrac{1}{4}\delta_\nu^\mu(\overline{\Psi}S^{\sigma\rho}\Psi)(\overline{\Psi}S_{\sigma\rho}\Psi)$$

which by means of the two identities (A.8) and (A.10) gives

$$T_\nu^\mu = \tfrac{1}{8}\left\{\delta_\nu^\mu\left[(\overline{\Psi}\Psi)^2 + (\overline{\Psi}\gamma^5\Psi)^2\right] - 2(\overline{\Psi}\gamma^\mu\Psi)(\overline{\Psi}\gamma_\nu\Psi) - 2(\overline{\Psi}\gamma^\mu\gamma^5\Psi)(\overline{\Psi}\gamma_\nu\gamma^5\Psi)\right\} \tag{2.24}$$

$$T_\nu^\mu T_\mu^\nu = \tfrac{1}{16}\left\{(\overline{\Psi}\Psi)^2 + (\overline{\Psi}\gamma^5\Psi)^2\right\}^2 \tag{2.25}$$

For a null field one therefore has

$$T_\mu^\nu T_\nu^\mu = 0 \tag{2.26}$$

if and only if one has the two equations

$$\overline{\Psi}\Psi = 0 \tag{2.27}$$

$$\overline{\Psi}\gamma^5\Psi = 0 \tag{2.28}$$

3. LORENTZ TRANSFORMATION

Let us consider an antisymmetric tensor $F^{\mu\nu}$ of the form (2.8), i.e.,

$$F^{\mu\nu} = \frac{i}{2}\overline{\Psi}\gamma^{[\mu}\gamma^{\nu]}\Psi$$

The transformation properties of the spinor Ψ under a Lorentz transformation are deduced by following Messiah (1966b, p. 904) quite thoroughly.

Since one has the anticommutation relations (2.7), we can write (2.8) as follows:

$$F^{\mu\nu} = i\overline{\Psi}\gamma^\mu\gamma^\nu\Psi \qquad (\mu \neq \nu) \tag{3.1}$$

Under a Lorentz transformation Ω, $F^{\mu\nu}$ transforms in $\bar{F}^{\mu\nu}$ according to

$$\bar{F}^{\mu\nu}=\Omega^{\mu}_{\sigma}\Omega^{\nu}_{\tau}F^{\sigma\tau} \tag{3.2}$$

i.e., from (3.1)

$$\bar{F}^{\mu\nu}=i\bar{\Psi}\hat{\gamma}^{\mu}\hat{\gamma}^{\nu}\Psi \qquad (\mu\neq\nu) \tag{3.3}$$

with

$$\hat{\gamma}^{\mu}=\Omega^{\mu}_{\sigma}\gamma^{\sigma} \tag{3.4}$$

But there exists a matrix Λ such that

$$\hat{\gamma}^{\mu}=\Lambda^{-1}\gamma^{\mu}\Lambda \tag{3.5}$$

with the property

$$\Lambda^{\dagger}=\gamma^{0}\Lambda^{-1}\gamma^{0} \tag{3.6}$$

or equivalently

$$\Lambda^{\dagger}\gamma^{0}=\gamma^{0}\Lambda^{-1} \tag{3.7}$$

so that equation (3.3), by means of definition (2.10) becomes

$$\bar{F}^{\mu\nu}=i\Psi^{\dagger}\gamma^{0}\hat{\gamma}^{\mu}\hat{\gamma}^{\nu}\Psi=i\Psi^{\dagger}\Lambda^{\dagger}\gamma^{0}\gamma^{\mu}\gamma^{\nu}\Lambda\Psi=\bar{\Psi}'\gamma^{\mu}\gamma^{\nu}\Psi' \tag{3.8}$$

where we have put

$$\Psi'=\Lambda\Psi \tag{3.9}$$

and equation (3.9) gives the transformation law for a spinor Ψ when the antisymmetric tensor (2.8) undergoes a Lorentz transformation.

4. THE RAINICH–MISNER–WHEELER DUALITY ROTATION AND THE TOUSCHECK–NISHIJIMA TRANSFORMATION

The effect of a duality transformation (2.11) on the spinor Ψ when an antisymmetric tensor (2.8) undergoes such a transformation is easily determined. By means of equation (2.18) the duality transformation (2.11) applied to the tensor (2.8) gives

$$\bar{F}^{\mu\nu}=\bar{\Psi}(\cos\alpha+\gamma^{5}\sin\alpha)S^{\mu\nu}\Psi, \tag{4.1}$$

and since equation (2.14) holds, we can write

$$e^{\gamma^5 \alpha} = \cos\alpha + \gamma^5 \sin\alpha \tag{4.2}$$

which permits one to write equation (4.1) as follows:

$$\bar{F}^{\mu\nu} = \bar{\Psi} e^{\gamma^5 \alpha} S^{\mu\nu} \Psi \tag{4.3}$$

It is easily shown that there exists a unitary matrix Δ such that

$$\gamma^0 e^{\gamma^5 \alpha} \gamma^\mu \gamma^\nu = \Delta^\dagger \gamma^0 \gamma^\mu \gamma^\nu \Delta \qquad \mu \neq \nu \tag{4.4}$$

Since γ^5 anticommutes with all γ's, i.e.,

$$\gamma^5 \gamma^\mu + \gamma^\mu \gamma^5 = 0 \tag{4.5}$$

by putting

$$\Delta = a + b\gamma^5 \tag{4.6}$$

with a and b two complex numbers, we have, for the anti-Hermitianity of γ^5 given by (2.15),

$$\Delta^\dagger = a^* - b^* \gamma^5 \tag{4.7}$$

By substituting into (4.4) for the anticommutation relation (4.2) and for the properties (2.17) and (4.2) one has

$$\cos\alpha + \gamma^5 \sin\alpha = |a|^2 - |b|^2 + (a^* b + a b^*)\gamma^5$$

which gives the system

$$\begin{aligned} |a|^2 - |b|^2 &= \cos\alpha \\ a^* b + a b^* &= \sin\alpha \end{aligned} \tag{4.8}$$

By assuming a and b complex, the system (45) is a system of two equations in four unknowns. However, it is easily shown that it admits a solution with a and b real. In fact in this hypothesis, the system (4.8) becomes

$$\begin{aligned} a - b &= \cos\alpha \\ 2ab &= \sin\alpha \end{aligned}$$

which admits the solution

$$a = \cos\frac{\alpha}{2}$$

$$b = \sin\frac{\alpha}{2}$$

in correspondence of which equation (4.3) gives

$$\Delta = \cos\frac{\alpha}{2} + \gamma^5 \sin\frac{\alpha}{2} = e^{\gamma^5\alpha/2} \tag{4.9}$$

We can therefore conclude that if an antisymmetric tensor in the form (2.8) undergoes a duality rotation with complexion α then the corresponding spinor Ψ undergoes a transformation, we call "spinor duality rotation," expressed by

$$\Psi \rightarrow e^{\gamma^5\alpha/2}\Psi \tag{4.10}$$

The transformation (4.10) was introduced in the theory of leptons by Touschek (1957) and in the same time by Nishijima (1957).

The unitary character of the spinor duality rotation transformation is readily seen once the anti-Hermitian character of the matrix γ^5 is taken into consideration, because, in fact, one has, for

$$\Delta = e^{\gamma^5\alpha}$$

$$\Delta^\dagger = e^{-\gamma^5\alpha}$$

hence the unitary property.

5. THE ELECTROMAGNETIC FIELD

The representability of a nonnull electromagnetic field in the spinor form (2.8) will be now immediately shown once it will be shown for an extremal field.

In fact, if for the extremal field $F_0^{\mu\nu}$ there exists an extremal spinor Ψ_0 such that (2.8) holds, i.e.,

$$F_0^{\mu\nu} = \frac{i}{2}\overline{\Psi}_0\gamma^{[\mu}\gamma^{\nu]}\Psi_0 \tag{5.1}$$

then for the Rainich–Misner–Wheeler theorem and for what we have demonstrated above, we have that any electromagnetic field $F^{\mu\nu}$ with complexion α can be written

$$F^{\mu\nu} = \frac{i}{2}\bar{\Psi}\gamma^{[\mu}\gamma^{\nu]}\Psi \tag{5.2}$$

with

$$\Psi = e^{\gamma^5\alpha/2}\Lambda\Psi_0 \tag{5.3}$$

In order to demonstrate the existence of a spinor Ψ_0 such that the representation (5.1) for an extremal field is valid, it is convenient to use a different representation of the Dirac matrices.

Since $i\gamma^1$ and $i\gamma^0\gamma^1\gamma^2$ are two Hermitian, commuting matrices, according to the well-known theorem which states that two commuting Hermitian matrices can be simultaneously diagonalized with the aid of the same unitary transformation, there will be a unitary matrix U such that both $Ui\gamma^1U^\dagger$ and $Ui\gamma^0\gamma^1\gamma^2U^\dagger$ are diagonal.

In the Dirac representation (2.16), both $\gamma^1\gamma^2$ and $\gamma^0\gamma^1\gamma^2$ are diagonal, and so by taking

$$U = \frac{1}{\sqrt{2}}(1-\gamma^2) \tag{5.4}$$

which is clearly unitary, i.e.,

$$UU^\dagger = U^\dagger U = I \tag{5.5}$$

one has

$$U\gamma^1U^\dagger = \gamma^1\gamma^2 \tag{5.6}$$

$$U\gamma^0\gamma^1\gamma^2U^\dagger = \gamma^0\gamma^1\gamma^2 \tag{5.7}$$

On the other hand since the new matrices γ'^μ defined by

$$\gamma'^\mu = U\gamma^\mu U^\dagger \tag{5.8}$$

satisfy the same anticommutation relations (2.7), we can assume, without loss of generality, the new representation (5.7) for showing the existence of a spinor $\hat{\Psi}_0$, such that (5.1) is satisfied.

By indicating ξ^μ ($\mu=0$–3) the components of the spinor $\hat{\Psi}_0$ in this representation, equation (5.1) explicitly reads

$$\rho_0^2-\rho_1^2=F_0^{12}+F_0^{01} \tag{5.9}$$

$$\rho_0\rho_1\cos(\theta_1-\theta_0)=-\tfrac{1}{2}(F_0^{30}+F_0^{23}) \tag{5.10}$$

$$\rho_2^2-\rho_3^2=F_0^{01}+F_0^{21} \tag{5.11}$$

$$\rho_2\rho_3\cos(\theta_3-\theta_2)=\tfrac{1}{2}(F_0^{30}+F_0^{32}) \tag{5.12}$$

$$\rho_0\rho_2\cos(\theta_2-\theta_0)+\rho_1\rho_3\cos(\theta_3-\theta_1)=F_0^{31} \tag{5.13}$$

$$\rho_0\rho_3\cos(\theta_3-\theta_0)-\rho_1\rho_2\cos(\theta_2-\theta_1)=F_0^{02} \tag{5.14}$$

Here, we have put

$$\xi^\mu=\rho_\mu e^{i\theta_\mu} \qquad (\mu=0-3) \tag{5.15}$$

On the other hand, $F_0^{\mu\nu}$ is the extremal field

$$F_0^{\mu\nu}=\begin{matrix} & 0 & 1 & 2 & 3 \\ 0 & 0 & E & 0 & 0 \\ 1 & -E & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{matrix} \tag{5.16}$$

and the system (5.9)–(5.14) reads

$$\rho_0^2-\rho_1^2=E \tag{5.17}$$

$$\rho_0\rho_1\cos(\theta_1-\theta_0)=0 \tag{5.18}$$

$$\rho_2^2-\rho_3^2=E \tag{5.19}$$

$$\rho_2\rho_3\cos(\theta_3-\theta_2)=0 \tag{5.20}$$

$$\rho_0\rho_2\cos(\theta_2-\theta_0)+\rho_1\rho_3\cos(\theta_3-\theta_1)=0 \tag{5.21}$$

$$\rho_0\rho_3\cos(\theta_3-\theta_0)-\rho_1\rho_2\cos(\theta_2-\theta_1)=0 \tag{5.22}$$

If $E > 0$, we can put

$$\rho_0 = \sqrt{E} \cosh \beta \tag{5.23}$$

$$\rho_1 = \sqrt{E} \sinh \beta \tag{5.24}$$

$$\rho_2 = \sqrt{E} \cosh \gamma \tag{5.25}$$

$$\rho_3 = \sqrt{E} \sinh \gamma \tag{5.26}$$

and the equations (5.17) and (5.19) are satisfied while the remaining equations become

$$\sinh 2\beta \cos(\theta_1 - \theta_0) = 0 \tag{5.27}$$

$$\sinh 2\gamma \cos(\theta_3 - \theta_2) = 0 \tag{5.28}$$

$$\cosh \beta \cosh \gamma \cos(\theta_2 - \theta_0) + \sinh \beta \sinh \gamma \cos(\theta_3 - \theta_1) = 0 \tag{5.29}$$

$$\cosh \beta \sinh \gamma \cos(\theta_3 - \theta_0) - \sinh \beta \cosh \gamma \cos(\theta_2 - \theta_1) = 0 \tag{5.30}$$

A simple algebraic analysis of this system, which we omit for sake of simplicity, leads, within a phase factor, to the following solution:

$$\hat{\Psi}_{0+} = \begin{pmatrix} \cosh\theta \\ i \sinh\theta \\ \varepsilon i \cosh\theta \\ \varepsilon \sinh\theta \end{pmatrix} \sqrt{E} \tag{5.31}$$

Here $\hat{\Psi}_{0+}$ indicates the spinor solution of equation (5.1) corresponding to positive values of the extremal field, E, θ is an arbitrary real parameter, and ε can assume values $+1$ and -1.

The solution $\hat{\Psi}_{0-}$ of equation (5.1) for negative values of the extremal field is readily found, since the transformation

$$\begin{aligned} &E \to -E \\ &\rho_0 \rightleftarrows \rho_1 \\ &\rho_2 \rightleftarrows \rho_3 \\ &\theta_0 \rightleftarrows \theta_1 \\ &\theta_2 \rightleftarrows \theta_3 \end{aligned} \tag{5.32}$$

reduces the case $E<0$ to the previous one, so that one has

$$\hat{\Psi}_{0-}=\begin{bmatrix} i\sinh\theta \\ \cosh\theta \\ \varepsilon\sinh\theta \\ \varepsilon i\cosh\theta \end{bmatrix}\sqrt{-E}=\begin{bmatrix} -\sinh\theta \\ i\cosh\theta \\ \varepsilon i\sinh\theta \\ -\varepsilon\cosh\theta \end{bmatrix}\sqrt{E} \tag{5.33}$$

The general solution of (5.1) therefore reads as follows:

$$\hat{\Psi}_0=\tfrac{1}{2}(1+\operatorname{sgn}E)\hat{\Psi}_{0+}+(1-\operatorname{sgn}E)\hat{\Psi}_{0-} \tag{5.34}$$

Here $\operatorname{sgn}x$ is the step function defined by

$$\operatorname{sgn}x=\begin{cases} +1 & \text{for } x>0 \\ -1 & \text{for } x<0 \end{cases} \tag{5.35}$$

The solution Ψ_0 of the system (5.1) in the Dirac representation is immediately found by inverting the transformation (5.8), and one has

$$\Psi_0=U^{\dagger}\hat{\Psi}_0=\frac{1}{\sqrt{2}}(1-\gamma^2)\hat{\Psi}_0 \tag{5.36}$$

The solution (5.36) is susceptible to a simpler representation. First we notice that by introducing the angle γ defined by

$$\gamma=\frac{1-\operatorname{sgn}E}{4}\pi \tag{5.37}$$

(5.34) reads

$$\hat{\Psi}_0=\cos\gamma\hat{\Psi}_{0+}+\sin\gamma\hat{\Psi}_{0-} \tag{5.38}$$

and since one readily has also

$$\hat{\Psi}_{0-}=\gamma^2\gamma^3\hat{\Psi}_{0+}=-i\sigma_x\hat{\Psi}_{0+} \tag{5.39}$$

equation (5.34) becomes

$$\hat{\Psi}_0=(\cos\gamma-i\sigma_x\sin\gamma)\hat{\Psi}_{0+}=\exp\left[i\sigma_x(\operatorname{sgn}E-1)\frac{\pi}{2}\right]\hat{\Psi}_{0+} \tag{5.40}$$

which shows that the spinor $\hat{\Psi}_0$ is obtained by rotating the spinor Ψ_{0+} about the axis $0x$ by an angle $(\operatorname{sgn}E-1)\pi/4$.

The spinor $\hat{\Psi}_{0+}$ also can be written in a simpler way by noticing that one has

$$\begin{bmatrix} \cosh\theta \\ i\sinh\theta \\ \varepsilon i\cosh\theta \\ \varepsilon\sinh\theta \end{bmatrix} = \begin{bmatrix} \cosh\theta & 0 & 0 & \varepsilon\sinh\theta \\ 0 & \cosh\theta & \varepsilon\sinh\theta & 0 \\ 0 & \varepsilon\sinh\theta & \cosh\theta & 0 \\ \varepsilon\sinh\theta & 0 & 0 & \cosh\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \varepsilon i \\ 0 \end{bmatrix} \tag{5.41}$$

or equivalently,

$$\begin{bmatrix} \cosh\theta \\ \sinh\theta \\ \varepsilon i\cosh\theta \\ \varepsilon\sinh\theta \end{bmatrix} = (\cosh\varepsilon\theta + \gamma^0\gamma^1\sinh\varepsilon\theta) \begin{bmatrix} 1 \\ 0 \\ \varepsilon i \\ 0 \end{bmatrix} \tag{5.42}$$

But one has (Messiah, 1966b, p. 906)

$$\cosh\varepsilon\theta + \gamma^0\gamma^1\sinh\varepsilon\theta = \Lambda^{xt}(-\varepsilon\theta) = e^{-\alpha_x\varepsilon\theta} \tag{5.43}$$

with

$$\gamma = \beta\alpha \qquad \text{and} \qquad \gamma^0 = \beta \tag{5.44}$$

and (5.41) and (5.42) show that the spinor Ψ_{0+} is obtained by the spinor

$$\begin{bmatrix} 1 \\ 0 \\ \varepsilon i \\ 0 \end{bmatrix}$$

by a special Lorentz transformation with velocity $v = \tanh(-2\varepsilon\theta)$ directed along the x axis (Messiah, 1966b, p. 906). The solution (5.36) can therefore be written as follows:

$$\Psi_0 = \left(\frac{E}{2}\right)^{1/2} (1-\gamma^2) e^{i\alpha_x(\operatorname{sgn} E - 1)\pi/2} e^{-\alpha_x\varepsilon\theta} \begin{bmatrix} 1 \\ 0 \\ \varepsilon i \\ 0 \end{bmatrix} \tag{5.45}$$

The existence of a solution of (5.1) for an extremal field we have constructed, implies, through equation (5.3), the existence of a solution of

(5.2), which then reads as follows:

$$\Psi = \left(\frac{E}{2}\right)^{1/2} e^{\gamma^5\alpha/2}\Lambda(1-\gamma^2)e^{i\sigma_x(\operatorname{sgn}E-1)\pi/2}e^{-\alpha_x\varepsilon\theta}\begin{bmatrix}1\\0\\\varepsilon i\\0\end{bmatrix} \tag{5.46}$$

We can therefore conclude that for any nonnull electromagnetic field $F^{\mu\nu}$ there exists a spinor Ψ such that

$$F^{\mu\nu} = \frac{i}{2}\overline{\Psi}\gamma^{[\mu}\gamma^{\nu]}\Psi$$

6. THE SPINOR FOR THE ZERO FIELD

While to a zero spinor corresponds a zero field, the reverse is not true. It is easily seen, for example, that the eigenvectors of γ^5 do indeed lead to a zero field. In fact, if Ψ is such that

$$\gamma^5\Psi = \varepsilon i\Psi \tag{6.1}$$

with $\varepsilon = \pm 1$, then

$$\overline{\Psi}\gamma^5 = -\varepsilon i\overline{\Psi} \tag{6.2}$$

and one has

$$\overline{\Psi}\gamma^5\gamma^{[\mu}\gamma^{\nu]}\Psi = -\varepsilon i\overline{\Psi}\gamma^{[\mu}\gamma^{\nu]}\Psi \tag{6.3}$$

On the other hand, since γ^5 anticommutes with the γ's one also has

$$\overline{\Psi}\gamma^5\gamma^{[\mu}\gamma^{\nu]}\Psi = \Psi\gamma^{[\mu}\gamma^{\nu]}\gamma^5\Psi \tag{6.4}$$

and equation (6.1) gives

$$\overline{\Psi}\gamma^5\gamma^{[\mu}\gamma^{\nu]}\Psi = \varepsilon i\overline{\Psi}\gamma^{[\mu}\gamma^{\nu]}\Psi \tag{6.5}$$

and its comparison with equation (6.3) leads to

$$\overline{\Psi}\gamma^{[\mu}\gamma^{\nu]}\Psi = 0 \tag{6.6}$$

It is readily seen that the eigenvectors of γ^5 give the whole set of solutions for the zero field. In fact, the system (5.9)–(5.14) for $F^{\mu\nu}=0$ gives

$$\begin{aligned}
&\rho_0=\rho_1\\
&\rho_2=\rho_3\\
&\rho_2^2\cos(\theta_1-\theta_0)=0\\
&\rho_2^2\cos(\theta_3-\theta_2)=0\\
&\rho_0\rho_2[\cos(\theta_2-\theta_1)+\cos(\theta_3-\theta_1)]=0\\
&\rho_0\rho_2[\cos(\theta_3-\theta_0)-\cos(\theta_2-\theta_1)]=0
\end{aligned} \tag{6.7}$$

which admits, beside the trivial solution, the following nontrivial solutions:

$$\xi\begin{bmatrix}0\\0\\1\\\varepsilon i\end{bmatrix},\quad \xi\begin{bmatrix}1\\\varepsilon i\\0\\0\end{bmatrix},\quad \begin{bmatrix}\xi\\\varepsilon i\xi\\\eta\\-\varepsilon i\eta\end{bmatrix} \tag{6.8}$$

with ξ and η arbitrary complex parameters and $\varepsilon=\pm1$. For having the zero-field solutions in the Dirac representation one has only to apply to the spinors (6.8) the transformation (5.4), which gives the following:

$$\xi\begin{bmatrix}-\varepsilon\\-i\\1\\\varepsilon i\end{bmatrix},\quad \xi\begin{bmatrix}1\\\varepsilon i\\\varepsilon\\i\end{bmatrix},\quad \begin{bmatrix}\xi+\varepsilon\eta\\i(\varepsilon\xi-\eta)\\\varepsilon(\xi+\varepsilon\eta)\\\varepsilon i(\varepsilon\xi-\eta)\end{bmatrix}=\begin{bmatrix}\sigma\\i\tau\\\varepsilon\sigma\\\varepsilon i\tau\end{bmatrix} \tag{6.9}$$

where in the third one $\xi+\varepsilon\eta$ and $\varepsilon\xi-\eta$ have been replaced with σ and τ. And it is readily seen that the solutions (6.9) are all eigenvectors of γ^5 corresponding to the eigenvalues εi.

7. THE SPINOR Ψ AND THE CHARGE CONJUGATION OPERATION

A property of the spinor Ψ which enters in the present spinor representation of the electromagnetic field is that it cannot be an eigenstate of the charge conjugation operator K_C (Messiah, 1966b, p. 916) with charge parity $+1$, or in other words it cannot be a neutrettor (Corson, 1955).

In order to demonstrate this property we can show it for the extremal field and then show that the property continues to hold after a Touschek–Nishijima transformation and a Lorentz transformation.

For the extremal field, we can assume, for sake of simplicity, that it is positive. In this hypothesis, the spinor Ψ reduces to $\hat{\Psi}_{0+}$ given by (5.31).

In the Dirac representation one has, from equation (5.36),

$$\Psi_{0+} = \frac{1}{\sqrt{2}}(1-\gamma^2)\hat{\Psi}_{0+} \tag{7.1}$$

and since in the same representation one also has

$$\Psi_C = \gamma^5 K \Psi, = \gamma^2 \Psi^* \tag{7.2}$$

(7.1) gives, by using equation (5.33),

$$\Psi_{0+C} = \frac{1}{\sqrt{2}}(1+\gamma^2)\hat{\Psi}_{0+C} = -\frac{1}{\sqrt{2}}(1-\gamma^2)\hat{\Psi}^*_{0+} \tag{7.3}$$

Therefore the condition

$$\Psi_{0+C} = \Psi_{0+} \tag{7.4}$$

is equivalent to the other

$$\hat{\Psi}_{0+} = \hat{\Psi}^*_{0+}$$

which cannot be satisfied since, from equation (5.31), it would require the incompatible condition

$$\cosh\theta = \sinh\theta = 0$$

and similarly for Ψ_{0-}.

For what concerns the Touschek–Nishijima transformation it is readily seen that it commutes with the charge conjugation operator because the operator K commutes with γ^μ then with γ^5. It is already known that the charge conjugation operator commutes with Lorentz transformations and so for the general spinor Ψ given by (5.3) one has

$$K_C \Psi = e^{\gamma^5 \alpha/2} \Lambda K \Psi_0$$

The spinor Ψ is then a self-charge conjugate if and only if Ψ_0 itself is such, and that is not possible, as has been shown above.

8. THE SPINOR EQUATION EQUIVALENT TO MAXWELL'S EQUATIONS

Maxwell's equations (2.22) and (2.23) will be now reduced to a single equation for the spinor Ψ. Since one has equation (2.19), by using the anticommutation relations (2.7), equations (2.22) and (2.23) read as follows:

$$\overline{\Psi}_{,\nu}\gamma^{\nu}\gamma_{\mu}\Psi-\overline{\Psi}\gamma_{\mu}\gamma^{\nu}\Psi_{,\nu}=\overline{\Psi}_{,\mu}\Psi-\overline{\Psi}\Psi_{,\mu}-2ij_{\mu} \tag{8.1}$$

becomes

$$\overline{\Psi}_{,\nu}\gamma^{\nu}\gamma^{5}\gamma_{\mu}\Psi-\overline{\Psi}\gamma_{\mu}\gamma^{5}\gamma^{\nu}\Psi_{,\nu}=-\left(\overline{\Psi}_{,\mu}\gamma^{5}\Psi-\overline{\Psi}\gamma^{5}\Psi_{,\mu}\right) \tag{8.2}$$

which by putting

$$\gamma^{\mu}\Psi_{,\nu}=\Phi \tag{8.3}$$

$$\overline{\Phi}\gamma_{\mu}\Psi-\overline{\Psi}\gamma_{\mu}\Phi=\overline{\Psi}_{,\mu}\Psi-\overline{\Psi}\Psi_{,\mu}-2ij_{\mu} \tag{8.4}$$

$$\overline{\Phi}\gamma^{5}\gamma_{\mu}\Psi-\overline{\Psi}\gamma_{\mu}\gamma^{5}\Phi=-\left(\overline{\Psi}_{,\mu}\gamma^{5}\Psi-\overline{\Psi}\gamma^{5}\Psi_{,\mu}\right) \tag{8.5}$$

Equations (8.4) and (8.5) can be solved for the spinor Φ by expanding Φ into the four independent eigenvectors of the matrix γ^5 corresponding to its eigenvalues $\pm i$, namely,

$$\chi_0=\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\quad \chi_1=\begin{pmatrix}1\\0\\-1\\0\end{pmatrix},\quad \chi_2=\begin{pmatrix}0\\1\\0\\1\end{pmatrix},\quad \chi_3=\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} \tag{8.6}$$

with

$$\gamma^5\chi_\nu=(-1)^{\nu+1}i\chi_\nu \tag{8.7}$$

By putting

$$\Phi=\Phi^{\nu}\chi_\nu \tag{8.8}$$

since one has the orthogonality property:

$$\tilde{\chi}_\mu\chi_\nu=2\delta_{\mu\nu} \tag{8.9}$$

where $\tilde{\chi}_\mu$ is the transposed of χ_μ, one has the components of any spinor Φ

in the representation χ, as follows:

$$\Phi_\nu = \tfrac{1}{2}\tilde{\chi}_\nu \Phi \tag{8.10}$$

By using the representation (8.8), and by remembering equation (8.7), equations (8.4) and (8.5) read

$$\Phi^{\nu *}\bar{\chi}_\nu \gamma_\mu \Psi - \Phi^\nu \bar{\Psi}\gamma_\mu \chi_\nu = \bar{\Psi}_{,\mu}\Psi - \bar{\Psi}\Psi_{,\mu} - 2ij_\mu \tag{8.11}$$

$$(-1)^\nu \Psi^{\nu *}\bar{\chi}_\nu \gamma_\nu \Psi + (-1)^\nu \Phi^\nu \bar{\Psi}\gamma_\mu \chi_\nu = i\left(\bar{\Psi}_{,\mu}\gamma^5\Psi - \bar{\Psi}\gamma^5\Psi_{,\mu}\right) \tag{8.12}$$

By adding and subtracting the last two equations one has

$$\Phi^{0*}\bar{\chi}_0\gamma_\mu\Psi - \Phi^1\bar{\Psi}\gamma_\mu\chi_1 + \Phi^{2*}\bar{\chi}_2\gamma_\mu\Psi - \Phi^3\bar{\Psi}\gamma_\mu\chi_3 = \bar{\Psi}_{,\mu}\frac{i\gamma^5+1}{2}\Psi - \bar{\Psi}\frac{i\gamma^5+1}{2}\Psi_{,\mu} - ij_\mu \tag{8.13}$$

$$\Phi^0\bar{\Psi}\gamma_\mu\bar{\chi}_0 - \Phi^{1*}\bar{\chi}_1\gamma_\mu\Psi + \Phi^2\bar{\Psi}\gamma_\mu\chi_2 - \Phi^3\bar{\chi}_3\gamma_\mu\Psi = \bar{\Psi}_{,\mu}\frac{i\gamma^5-1}{2}\Psi - \bar{\Psi}\frac{i\gamma^5-1}{2}\Psi_{,\mu} + ij_\mu \tag{8.14}$$

Since, as is readily seen, equation (8.14) is nothing but the complex conjugate of equation (8.13) we have to consider only this latter equation, which by putting

$$\Gamma_\mu = \bar{\Psi}_{,\mu}\frac{1+i\gamma^5}{2}\Psi - \bar{\Psi}\frac{1+i\gamma^5}{2}\Psi_{,\mu} - ij_\mu \tag{8.15}$$

reads

$$\Phi^{0*}\bar{\chi}_0\gamma_\mu\Psi - \Phi^1\bar{\Psi}\gamma_\mu\bar{\chi}_1 + \Phi^{2*}\bar{\chi}_2\gamma_\mu\Psi - \Phi^3\bar{\Psi}\gamma_\mu\chi_3 = \Gamma_\mu \tag{8.16}$$

If η^ν are the components of the spinor Ψ in the representation of the χ's, namely,

$$\Psi = \eta^\nu \chi_\nu \tag{8.17}$$

i.e., according to equation (8.10).

$$\eta^\nu = \tfrac{1}{2}\tilde{\chi}_\nu \Psi \tag{8.18}$$

Straightforward calculation of the coefficients of (8.16) gives the following:

$$
\begin{aligned}
&\bar{\chi}_0\gamma_0\Psi=2\eta_0, \quad \bar{\chi}_1\gamma_0\Psi=2\eta_1, \quad \bar{\chi}_2\gamma_0\Psi=2\eta_2, \quad \bar{\chi}_3\gamma_0\Psi=2\eta_3\\
&\bar{\chi}_0\gamma_1\Psi=-2\eta_2, \quad \bar{\chi}_1\gamma_1\Psi=2\eta_3, \quad \bar{\chi}_2\gamma_1\Psi=-2\eta_0, \quad \bar{\chi}_3\gamma_1\Psi=2\eta_1\\
&\bar{\chi}_0\gamma_{12}\Psi=2i\eta_2, \quad \bar{\chi}_1\gamma_2\Psi=-2i\eta_3, \quad \bar{\chi}_2\gamma_2\Psi=-2i\eta_0, \quad \bar{\chi}_3\gamma_2\Psi=2i\eta_1\\
&\bar{\chi}_0\gamma_3\Psi=-2\eta_0, \quad \bar{\chi}_1\gamma_3\Psi=2\eta_1, \quad \bar{\chi}_2\gamma_3\Psi=2\eta_2, \quad \bar{\chi}_3\gamma_3\Psi=-2\eta_3
\end{aligned}
\tag{8.19}
$$

the system (8.16) explicitly reads

$$
2\begin{pmatrix} \eta_0 & -\eta_1^* & \eta_2 & -\eta_3^* \\ -\eta_2 & -\eta_3^* & -\eta_0 & -\eta_1^* \\ i\eta_2 & -\eta_3^* & -i\eta_0 & i\eta_1^* \\ -\eta_0 & -\eta_1^* & \eta_2 & \eta_3^* \end{pmatrix} \begin{pmatrix} \Phi^{0*} \\ \Phi^1 \\ \Phi^{2*} \\ \Phi^3 \end{pmatrix} = \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{pmatrix} \tag{8.20}
$$

The latter system can be simplified by multiplying it on the left by the matrix Λ given by

$$
\Lambda=\frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & i & 0 \\ 0 & -1 & -i & 0 \end{pmatrix} \tag{8.21}
$$

and it reduces to

$$
AK=\Lambda\Gamma \tag{8.22}
$$

with

$$
K=\begin{pmatrix} \Phi_0^* \\ \Phi_1 \\ \Phi_2^* \\ \Phi_3 \end{pmatrix}, \qquad \Gamma=\begin{pmatrix} \Gamma_0 \\ \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{pmatrix} \tag{8.23}
$$

and

$$A=\begin{bmatrix} \eta_0 & 0 & 0 & -\eta_3^* \\ 0 & \eta_1^* & -\eta_2 & 0 \\ 0 & \eta_3^* & \eta_0 & 0 \\ \eta_2 & 0 & 0 & \eta_1^* \end{bmatrix} \tag{8.24}$$

It is readily seen that the matrix A is nonsingular. In fact,

$$\det A=(\eta_0\eta_1^*+\eta_2\eta_3^*)^2 \tag{8.25}$$

and by using equation (8.18) one straightforwardly has

$$\eta_0\eta_1^*+\eta_2\eta_3^*=\tfrac{1}{4}(\bar{\Psi}\Psi+i\bar{\Psi}\gamma^5\Psi) \tag{8.26}$$

So $\det A$ would be zero if and only if $\bar{\Psi}\Psi=0$ and $\bar{\Psi}\gamma^5\Psi=0$, which are the conditions (2.27) and (2.28) for the null field. We are in the hypothesis of a nonnull field. A^{-1} does, therefore, exist and it is immediately found to be

$$A^{-1}=\frac{1}{\eta_0\eta_1^*+\eta_2\eta_3^*}\begin{bmatrix} \eta_1^* & 0 & 0 & \eta_3^* \\ 0 & \eta_0 & \eta_2 & 0 \\ 0 & -\eta_3^* & \eta_1 & 0 \\ -\eta_2 & 0 & 0 & \eta_0 \end{bmatrix} \tag{8.27}$$

and (8.22) gives

$$K=A^{-1}\Lambda\Gamma \tag{8.28}$$

By taking in account the contravariant representation of (8.19) equation (8.28) explicitly reads

$$K=\frac{1}{8(\eta_0\eta_1^*+\eta_2\eta_3^*)}\begin{bmatrix} \bar{\Psi}\gamma^0\chi_1 & \bar{\Psi}\gamma^1\chi_1 & \bar{\Psi}\gamma^2\chi_1 & \bar{\Psi}\gamma^3\chi_1 \\ -\bar{\chi}_0\gamma^0\Psi & -\bar{\chi}_0\gamma^1\Psi & -\bar{\chi}_0\gamma^2\Psi & -\bar{\chi}_0\gamma^3\Psi \\ \bar{\Psi}\gamma^0\chi_3 & \bar{\Psi}\gamma^1\chi_3 & \bar{\Psi}\gamma^2\chi_3 & \bar{\Psi}\gamma^3\chi_3 \\ -\bar{\chi}_2\gamma^0\Psi & -\bar{\chi}_2\gamma^1\Psi & -\bar{\chi}_2\gamma^2\Psi & -\bar{\chi}_2\gamma^3\Psi \end{bmatrix}\Gamma \tag{8.29}$$

On the other hand the vector (8.15) can be written

$$\Gamma_\mu=R_\mu+I_\mu \tag{8.30}$$

with

$$R_\mu = \tfrac{1}{2}\left(\overline{\Psi}_{,\mu} i\gamma^5\Psi - \overline{\Psi} i\gamma^5\Psi_{,\mu}\right) = -\operatorname{Im}\left(\overline{\Psi}_{,\mu}\gamma^5\Psi\right) \tag{8.31}$$

and

$$I_\mu = \tfrac{1}{2}\left(\overline{\Psi}_{,\mu}\Psi - \overline{\Psi}\Psi_{,\mu}\right) - ij_\mu = i\left[\operatorname{Im}\left(\overline{\Psi}_{,\mu}\Psi\right) - j_\mu\right] \tag{8.32}$$

where Im is the imaginary part, and for (8.23) the system (8.29) reads

$$\Phi^{0*} = \frac{1}{8(\eta_0\eta_1 + \eta_2\eta_3^*)}\left[\bar{\chi}_1(\gamma^\mu R_\mu - \gamma^\mu I_\mu)\Psi\right]^* \tag{8.33}$$

$$\Phi^1 = \frac{-1}{8(\eta_0\eta_1^* + \eta_2\eta_3^*)}\left[\bar{\chi}_0(\gamma^\mu R_\mu + \gamma^\mu I_\mu)\Psi\right] \tag{8.34}$$

$$\Phi^{2*} = \frac{1}{8(\eta_0\eta_1^* + \eta_2\eta_3^*)}\left[\bar{\chi}_3(\gamma^\mu R_\mu - \gamma^\mu I_\mu)\Psi\right]^* \tag{8.35}$$

$$\Phi^3 = \frac{-1}{8(\eta_0\eta_1^* + \eta_2\eta_3^*)}\left[\bar{\chi}_2(\gamma^\mu R_\mu + \gamma^\mu I_\mu)\Psi\right] \tag{8.36}$$

On the other hand since one clearly has

$$\bar{\chi}_1 = \tilde{\chi}_0, \qquad \bar{\chi}_0 = \tilde{\chi}_1, \qquad \bar{\chi}_3 = \tilde{\chi}_2, \qquad \bar{\chi}_2 = \tilde{\chi}_3 \tag{8.37}$$

and (8.7) holds, the system (8.33)–(8.36) becomes

$$4(\eta_0\eta_1^* + \eta_2\eta_3^*)^*\Phi^0 = -\tfrac{1}{2}\tilde{\chi}_0\gamma^\mu\left(i\gamma^5 R_\mu + I_\mu\right)\Psi \tag{8.38}$$

$$4(\eta_0\eta_1^* + \eta_2\eta_3^*)\Phi^1 = -\tfrac{1}{2}\tilde{\chi}_1\gamma^\mu\left(i\gamma^5 R_\mu + I_\mu\right)\Psi \tag{8.39}$$

$$4(\eta_0\eta_1^* + \eta_2\eta_3^*)^*\Phi^2 = -\tfrac{1}{2}\tilde{\chi}_2\gamma^\mu\left(i\gamma^5 R_\mu + I_\mu\right)\Psi \tag{8.40}$$

$$4(\eta_0\eta_1^* + \eta_2\eta_3^*)\Phi^3 = -\tfrac{1}{2}\tilde{\chi}_3\gamma^\mu\left(i\gamma^5 R_\mu + I_\mu\right)\Psi \tag{8.41}$$

By using (8.26) it can be easily shown that the left sides of the system (8.38)–(8.41) are nothing but the χ components of the spinor $C\Phi$ with the matrix C given by

$$C = \overline{\Psi}\Psi + \gamma^5\overline{\Psi}\gamma^5\Psi \tag{8.42}$$

and one has

$$\Phi = -C^{-1}\gamma^\mu\left(i\gamma^5 R_\mu + I_\mu\right)\Psi \tag{8.43}$$

The matrix C^{-1} is readily calculated to be the following:

$$C^{-1} = \frac{\overline{\Psi}\Psi - \gamma^5\overline{\Psi}\gamma^5\Psi}{(\overline{\Psi}\Psi)^2 + (\overline{\Psi}\gamma^5\Psi)^2} \tag{8.44}$$

Equations (8.31), (8.32) and (8.44) permit one to write explicitly the spinor Φ given by (8.43) as follows:

$$\Phi = -i\gamma^\mu \frac{\overline{\Psi}\Psi + \gamma^5\overline{\Psi}\gamma^5\Psi}{(\overline{\Psi}\Psi)^2 + (\overline{\Psi}\gamma^5\Psi)^2}\left\{\mathrm{Im}\left(\overline{\Psi}_{,\mu}\Psi\right) - \gamma^5\,\mathrm{Im}\left(\overline{\Psi}_{,\mu}\gamma^5\Psi\right) - j_\mu\right\}\Psi \tag{8.45}$$

It is just enough to remember equation (8.3) for obtaining the spinor equation equivalent to the system of Maxwell's equations, i.e.,

$$\gamma^\nu\Psi_{,\nu} = -i\gamma^\mu \frac{\overline{\Psi}\Psi + \gamma^5\overline{\Psi}\gamma^5\Psi}{(\overline{\Psi}\Psi)^2 + (\overline{\Psi}\gamma^5\Psi)^2}\left\{\mathrm{Im}\left(\overline{\Psi}_{,\mu}\Psi\right) - \gamma^5\,\mathrm{Im}\left(\overline{\Psi}_{,\mu}\gamma^5\Psi\right) - j_\mu\right\}\Psi \tag{8.46}$$

By introducing the angle α defined by

$$\frac{\overline{\Psi}\Psi}{\left[(\overline{\Psi}\Psi)^2 + (\overline{\Psi}\gamma^5\Psi)^2\right]^{1/2}} = \cos\alpha \tag{8.47}$$

$$\frac{\overline{\Psi}\gamma^5\Psi}{\left[(\overline{\Psi}\Psi)^2 + (\overline{\Psi}\gamma^5\Psi)^2\right]^{1/2}} = \sin\alpha \tag{8.48}$$

and by recalling equation (3.8), equation (8.46) can be written as follows:

$$\gamma^\mu\Psi_{,\nu} = -i\gamma^\mu \frac{e^{\gamma^5\alpha}}{\left[(\overline{\Psi}\Psi)^2 + (\overline{\Psi}\gamma^5\Psi)^2\right]^{1/2}}\left\{\mathrm{Im}\left(\overline{\Psi}_{,\mu}\Psi\right) - \gamma^5\,\mathrm{Im}\left(\overline{\Psi}_{,\mu}\gamma^5\Psi\right) - j_\mu\right\}\Psi \tag{8.49}$$

This angle α enters therefore in the spinor Maxwell equation through a duality rotation. The meaning of this angle will appear clearly in the next section.

9. THE MEANING OF THE ANGLE α

It will be now shown that the angle α we have just encountered in the spinor Maxwell equation of the previous section is nothing but the complexion of the electromagnetic field.

From equation (8.47) and (8.48) one has

$$\tan\alpha = \frac{\bar{\Psi}\gamma^5\Psi}{\bar{\Psi}\Psi} \tag{9.1}$$

which gives

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{(\bar{\Psi}\Psi)(\bar{\Psi}\gamma^5\Psi)}{1/2\left\{(\bar{\Psi}\Psi)^2-(\bar{\Psi}\gamma^5\Psi)^2\right\}} \tag{9.2}$$

By using the identities (A.10) and (A.11) together with equations (2.20) and (2.21), (9.2) reads

$$\tan 2\alpha = \frac{F_{\mu\nu}{}^* F^{\mu\nu}}{F_{\mu\nu}F^{\mu\nu}} \tag{9.3}$$

which coincides with the Misner–Wheeler equation (2.24) (Misner and Wheeler, 1957), when the angle α coincides with the complexion of the electromagnetic field $F_{\mu\nu}$.

10. CONCLUSIONS

In the previous pages, it has been shown that for any given electromagnetic field its electromagnetic field tensor $F^{\mu\nu}$ can be written as follows:

$$F^{\mu\nu} = \bar{\Psi}S^{\mu\nu}\Psi \tag{2.20}$$

where Ψ is a spinor, $\bar{\Psi}$ its Dirac conjugate, and $S^{\mu\nu}$ the spin operator given by

$$S^{\mu\nu} = \frac{i}{2}\gamma^{[\mu}\gamma^{\nu]} \tag{2.19}$$

the γ's being the Dirac matrices. In this representation Maxwell's equations read as follows:

$$(\overline{\Psi} S^{\mu\nu}\Psi)_{,\mu} = j^{\nu} \tag{2.22}$$

$$(\overline{\Psi}\gamma^5 S^{\mu\nu}\Psi)_{,\mu} = 0 \tag{2.23}$$

It has been shown, moreover, that the Rainich–Misner–Wheeler duality rotation of the "already unified theory" is nothing but the Touschek–Nishijima transformation of the theory of leptons. The two equations (2.22) and (2.23), quadratic in Ψ, have been reduced to a single nonlinear equation for Ψ which is the following:

$$\gamma^{\nu}\Psi_{,\nu} = -i\gamma^{\mu}\frac{e^{\gamma^5\alpha}}{\left[(\overline{\Psi}\Psi)^2 + (\overline{\Psi}\gamma^5\Psi)^2\right]^{1/2}}\left\{\mathrm{Im}\left(\overline{\Psi}_{,\mu}\Psi\right) - \gamma^5\,\mathrm{Im}\left(\overline{\Psi}_{,\mu}\gamma^5\Psi\right) - j_{\mu}\right\}\Psi \tag{8.49}$$

and the parameter α has been identified with the "complexions" of the electromagnetic field. The present paper is the first of a series in which the properties of the spinor representation of Maxwell's equations are explored and their implications with relativistic quantum mechanics analyzed in detail.

APPENDIX A: PROOF OF SOME IDENTITIES USED IN THE TEXT

In this Appendix are deduced some identities which are used in the text. The first one is the following:

$$(\overline{\Phi}\gamma^{\mu}\Psi)(\overline{\Phi}\gamma_{\mu}\Psi) = (\overline{\Phi}\Psi)^2 + (\overline{\Phi}\gamma^5\Psi)^2 \tag{A.1}$$

for any two spinors Φ and Ψ. Its demonstration is, however, omitted since it is only a tedious exercise. In particular for $\Phi = \Psi$ one has

$$(\overline{\Psi}\gamma^{\mu}\Psi)(\overline{\Psi}\gamma_{\mu}\Psi) = (\overline{\Psi}\Psi)^2 + (\overline{\Psi}\gamma^5\Psi)^2 \tag{A.2}$$

In what follows is used the following important identity:

$$(\overline{\Phi}_1\gamma^{\mu}\Psi)(\overline{\Phi}_2\gamma_{\mu}\Psi) = (\overline{\Phi}_1\Psi)(\overline{\Phi}_2\Psi) + (\overline{\Phi}_1\gamma^5\Psi)(\overline{\Phi}_2\gamma^5\Psi) \tag{A.3}$$

for any three spinors Φ_1, Φ_2, and Ψ, and which generalizes the identity (A.1). In fact, by putting

$$\Phi = \Phi_1 + \Phi_2 \tag{A.4}$$

the identity (A.1) gives

$$\left[\left(\overline{\Phi}_1 + \overline{\Phi}_2\right)\gamma^\mu \Psi\right]\left[\left(\overline{\Phi}_1 + \overline{\Phi}_2\right)\gamma_\mu \Psi\right]$$

$$= \left(\overline{\Phi}_1 \gamma^\mu \Psi\right)\left(\overline{\Phi}_1 \gamma_\mu \Psi\right) + \left(\overline{\Phi}_2 \gamma^\mu \Psi\right)\left(\overline{\Phi}_2 \gamma_\mu \Psi\right) + 2\left(\overline{\Phi}_1 \gamma^\mu \Psi\right)\left(\overline{\Phi}_2 \gamma_\mu \Psi\right) \tag{A.5}$$

It is just enough to use the identify (A.1) on the left side of (A.5) as well as on the first and second terms of the right side for having the identity (A.3). In particular, by taking in (A.3)

$$\Phi_2 = \gamma^5 \Phi_1 = \gamma^5 \Phi \tag{A.6}$$

(A.3) gives

$$\left(\overline{\Phi}\gamma^\mu \Psi\right)\left(\overline{\Phi}\gamma^5 \gamma_\mu \Psi\right) = 0 \tag{A.7}$$

Another important identity is the following:

$$\left(\overline{\Psi} S^{\mu\nu} \Psi\right)\left(\overline{\Psi} S_{\sigma\nu} \Psi\right) = \tfrac{1}{4}\left\{\delta^\mu_\sigma \left(\overline{\Psi}\Psi\right)^2 - \left(\overline{\Psi}\gamma^\mu \Psi\right)\left(\overline{\Psi}\gamma_\sigma \Psi\right) - \left(\overline{\Psi}\gamma^\mu \gamma^5 \Psi\right)\left(\overline{\Psi}\gamma_\sigma \gamma^5 \Psi\right)\right\} \tag{A.8}$$

which is immediately deduced from (A.3) by remembering (5.7) and the anticommutation relations (4.5) and by writing (A.3) with the positions

$$\Phi_1 = \gamma^\mu \Psi \qquad \text{and} \qquad \Phi_2 = \gamma_\sigma \Psi \tag{A.9}$$

From (A.8) and (A.2) one has

$$\left(\overline{\Psi} S^{\mu\nu} \Psi\right)\left(\overline{\Psi} S_{\mu\nu} \Psi\right) = \tfrac{1}{2}\left\{\left(\overline{\Psi}\Psi\right)^2 - \left(\overline{\Psi}\gamma^5 \Psi\right)^2\right\} \tag{A.10}$$

Similarly one has the other identity

$$\left(\overline{\Psi}\gamma^5 S^{\mu\nu} \Psi\right)\left(\overline{\Psi} S_{\mu\nu} \Psi\right) = \left(\overline{\Psi}\Psi\right)\left(\overline{\Psi}\gamma^5 \Psi\right) \tag{A.11}$$

Also the following identity holds:

$$(\bar{\Psi}\gamma^{\mu}\Psi)(\bar{\Psi}S_{\mu\nu}\Psi)=\frac{i}{2}(\bar{\Psi}\gamma^{5}\Psi)(\bar{\Psi}\gamma^{5}\gamma_{\nu}\Psi) \tag{A.12}$$

whose validity is easily shown by adopting the techniques used for the other identities. Another important identity is the following:

$$2i\,\mathrm{Im}\{(\bar{\Psi}_{,\nu}\Psi)(\bar{\Psi}\gamma^{5}\Psi)-(\bar{\Psi}_{,\nu}\gamma^{5}\Psi)(\bar{\Psi}\Psi)\}=(\bar{\Psi}\gamma^{\mu}\Psi)_{,\nu}(\bar{\Psi}\gamma_{\mu}\gamma^{5}\Psi) \tag{A.13}$$

whose demonstration is straightforward. In fact, the identity (A.3) with the positions

$$\Phi_1=\Psi_{,\nu} \tag{A.14}$$

$$\Phi_2=\gamma^{5}\Psi \tag{A.15}$$

gives

$$(\bar{\Psi}_{,\nu}\gamma^{\mu}\Psi)(\bar{\Psi}\gamma^{5}\gamma_{\mu}\Psi)=(\bar{\Psi}_{,\nu}\Psi)(\bar{\Psi}\gamma^{5}\Psi)-(\bar{\Psi}_{,\nu}\gamma^{5}\Psi)(\bar{\Psi}\Psi) \tag{A.16}$$

so that

$$\begin{aligned}2i\,\mathrm{Im}\{(\bar{\Psi}_{,\nu}\Psi)(\bar{\Psi}\gamma^{5}\Psi)-(\bar{\Psi}_{,\nu}\gamma^{5}\Psi)(\bar{\Psi}\Psi)\}\\ &=(\bar{\Psi}_{,\nu}\gamma^{\mu}\Psi)(\bar{\Psi}\gamma^{5}\gamma_{\mu}\Psi)-(\bar{\Psi}\gamma^{\mu}\Psi_{,\nu})(\bar{\Psi}\gamma_{\mu}\gamma^{5}\Psi)\\ &=\{(\bar{\Psi}_{,\nu}\gamma^{\mu}\Psi)+(\bar{\Psi}\gamma^{\mu}\Psi_{,\nu})\}(\bar{\Psi}\gamma_{\mu}\gamma^{5}\Psi)\end{aligned} \tag{A.17}$$

which is the identity (A.13). Moreover since one has the identity (A.7), (A.13) is equivalent to the other:

$$2i\,\mathrm{Im}\{(\bar{\Psi}_{,\nu}\Psi)(\bar{\Psi}\gamma^{5}\Psi)-(\bar{\Psi}_{,\nu}\gamma^{5}\Psi)(\bar{\Psi}\Psi)\}=-(\bar{\Psi}\gamma^{\mu}\Psi)(\bar{\Psi}\gamma_{\mu}\gamma^{5}\Psi)_{,\nu} \tag{A.18}$$

APPENDIX B: PROOF OF NONEXISTENCE OF A SPINOR Ψ SUCH THAT $\bar{\Psi}\gamma^\mu\Psi = j^\mu$ FOR ANY GIVEN VECTOR j^μ

For any given vector j_μ ($\mu = 0$–3), suppose there exists a spinor with components ξ^μ such that one has

$$\bar{\Psi}\gamma^\mu\Psi = j^\mu \tag{B.1}$$

It is easily shown that the system (B.1) is in general incompatible. By using the representation (4.7) for the matrices γ, the system (B.1) explicitly reads

$$\begin{aligned}
&\xi_0^*\xi_0 + \xi_1^*\xi_1 + \xi_2^*\xi_2 + \xi_3^*\xi_3 = j_0 \\
&\xi_3^*\xi_0 + \xi_2^*\xi_1 + \xi_1^*\xi_3 + \xi_0^*\xi_3 = j_1 \\
&\xi_3^*\xi_0 - \xi_2^*\xi_1 + \xi_1^*\xi_2 - \xi_0^*\xi_3 = -ij_2 \\
&\xi_2^*\xi_0 - \xi_3^*\xi_1 + \xi_0^*\xi_2 - \xi_1^*\xi_3 = j_3
\end{aligned} \tag{B.2}$$

or equivalently

$$A^*\Psi = J \tag{B.3}$$

with

$$A = \begin{bmatrix} \xi_0 & \xi_1 & \xi_2 & \xi_3 \\ \xi_3 & \xi_2 & \xi_1 & \xi_0 \\ \xi_3 & -\xi_2 & \xi_1 & -\xi_0 \\ \xi_2 & -\xi_3 & \xi_0 & -\xi_1 \end{bmatrix} \tag{B.4}$$

and

$$J = \begin{bmatrix} j_0 \\ j_1 \\ -ij_2 \\ j_3 \end{bmatrix} \tag{B.5}$$

The incompatibility of the system (B.2) is readily shown since a straightforward evaluation of $\det A$ gives

$$\det A = 0 \tag{B.6}$$

for any arbitrary spinor Ψ.

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